4. Confidence intervals

“95% of statistics is made up on the spot”

Anonymous
Confidence interval for count data
Confidence interval for count data

- Standard error of a count, $C$, is
  \[ SE = \sqrt{C} \]
- For example $5 \pm 2$ (after rounding up)
- How to find a confidence interval on $\mu$?
- Exact method: a bit complicated

$$C = 5 \pm 2 \text{ (SE)}$$
Confidence interval for count data: hand waving

Find $\mu_1$ such that $k$ cuts off 2.5% at right

$\mu_1 = 1.62$

$k = 5$

Find $\mu_2$ such that $k$ cuts off 2.5% at left

$\mu_2 = 11.67$

$C = 5 \pm 2$ (SE)

$C = 5^{+7}_{-3}$ (95% CI)
Confidence interval for count data: exact method

- Solving equations for Poisson cumulative distribution

```r
> poisson.test(5, conf.level = 0.95)

    Exact Poisson test

data:  5 time base: 1
number of events = 5, time base = 1, p-value = 0.00366
alternative hypothesis: true event rate is not equal to 1
95 percent confidence interval:
  1.623486 11.668332
sample estimates:
  event rate
  5
```
Count errors: example
Confidence intervals for count data are not integer

- 95% CI for $C = 5$ is $[1.6, 11.8]$
- Shouldn’t the confidence interval be exactly integer?
- Confidence interval is not for the sample count!
- We expect the true mean to be within $[1.6, 11.8]$ with a certain confidence
- The mean in a Poisson process is not integer
- Confidence intervals are for the true mean and are not integer
Confidence interval of the correlation coefficient
Confidence interval of the correlation coefficient

- Pearson’s correlation coefficient $r$ for a sample of pairs $(x_i, y_i)$
- It is a number between -1 and 1

- It is not enough to say “we find $r = 0.82$, therefore our samples are correlated”

- Confidence limits on $r$ or significance of correlation
Sampling distribution of the correlation coefficient

- **Gedankenexperiment**
- Consider a population of pairs of numbers \((x_i, y_i)\)
- The (unknown) population correlation coefficient, \(\rho = 0.7\)

- Draw lots of samples of pairs, size \(n\)
- Calculate the correlation coefficient for each sample
- Build a sampling distribution of the correlation coefficient
Sampling distribution of the correlation coefficient

- Sampling distribution of $r$
- Unknown in analytical form

- Let us transform it into a known distribution

- Fisher’s transformation:
  \[
  Z = \frac{1}{2} \ln \frac{1 + r}{1 - r}
  \]

- Build a sampling distribution of $Z$
Confidence interval of the correlation coefficient

\[ Z = \frac{1}{2} \ln \frac{1 + r}{1 - r} \]

Sampling distribution of \( r \)

Sampling distribution of \( Z \)

\[ \mu = \hat{Z}, \quad \sigma = \frac{1}{\sqrt{n - 3}} \]
Example: 95% confidence limits on $r$

- $n = 30$ and $r = 0.7$
- First, find

$$Z = \frac{1}{2} \ln \frac{1 + r}{1 - r} = 0.867$$

$$\sigma = \frac{1}{\sqrt{n - 3}} = 0.192$$

- $Z$ is normally distributed
- 95% CI corresponds to $Z \pm 1.96\sigma$:
  - $Z_L = Z - 1.96\sigma = 0.490$
  - $Z_U = Z + 1.96\sigma = 1.24$
Example: 95% confidence limits on $r$

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- $Z$ is normally distributed
- 95% CI corresponds to $Z \pm 1.96\sigma$:
  - $Z_L = Z - 1.96\sigma = 0.490$
  - $Z_U = Z + 1.96\sigma = 1.24$
- Now we find the corresponding limits on $r$
  
  $$r = \frac{e^{2Z} - 1}{e^{2Z} + 1}$$
  - $r_L = 0.454$
  - $r_U = 0.847$

- Hence, with 95% confidence, $r = 0.7^{\pm 0.15}$
How to do this in R

```r
> r <- 0.7
> n <- 30
> Z <- 0.5 * log((1+r) / (1-r))
> Z
[1] 0.8673005
> sigma <- 1 / sqrt(n - 3)
> sigma
[1] 0.1924501
> Z95 <- qnorm(0.975)
> Z95
[1] 1.959964
> Z.limits <- c(Z - Z95 * sigma, Z + Z95 * sigma)
> Z.limits
[1] 0.4901053 1.2444958
> r.limits <- (exp(2*Z.limits) - 1) / (exp(2*Z.limits) + 1)
> r.limits
[1] 0.4543000 0.8467329
```
# generate random data (in reproducible way)
> set.seed(47)
> x <- 1:30
> y <- x + rnorm(30, 0, 7)
# correlation test to find CI
> cor.test(x, y)

Pearson's product-moment correlation

data:  x and y
t = 5.1419, df = 28, p-value = 1.882e-05
alternative hypothesis: true correlation is not equal to 0
95 percent confidence interval:
  0.4494788 0.8450094
sample estimates:
   cor
0.6968971
Example: 95% CI for correlation with $n = 6$ and $n = 30$

$r = 0.7$

$CI = [-0.26, 0.96]$  
$p = 0.12$

$CI = [0.45, 0.85]$  
$p = 2 \times 10^{-5}$
Confidence interval of a proportion
Confidence interval of a proportion

- Proportion:
  \[ \hat{p} = \frac{\hat{S}}{n} = \frac{\text{number of successes}}{\text{sample size}} \]

- Examples:
  - poll results
  - survival experiments
  - counting cells with a property

- Sample proportion, \( \hat{p} \), is an estimator of the (unknown) population proportion, \( p \)

  \[
  \hat{S} = 4 \\
  n = 12 \\
  \hat{p} = \frac{4}{12} = 0.33
  \]
Sampling distribution of a proportion

- **Gedankenexperiment**
  - Consider a population of mice where $p = 13\%$ are immune to a certain disease

- Draw a random sample of size $n$ and find the proportion of immune mice, $\hat{p}$, in the sample

- Repeat 100,000 times and plot the distribution of $\hat{p}$

- What kind of distribution is it?
  - Hint: every time you select a mouse, it can be either immune or not, with probability $p$ or $1 - p$

- Binomial distribution
  - immune = “success”, probability $p$
  - not immune = “failure”, probability $1 - p$

- Good! Sampling distribution is known
Sampling distribution of a proportion: scaled binomial

**Absolute numbers**
- $X$ – binomial random variable
- Mean and standard deviation
  \[
  \mu = np \\
  \sigma = \sqrt{np(1-p)}
  \]

**Proportion**
- $R = X/n$ – scaled binomial random variable
- Mean and standard deviation scaled by $n$:
  \[
  \mu_R = p \\
  \sigma_R = \sqrt{\frac{p(1-p)}{n}}
  \]
Sampling distribution of a proportion

- Width of the sampling distribution of a proportion

\[ \sigma_R = \sqrt{\frac{p(1-p)}{n}} \]
Standard error of the mean

Hypothetical experiment
- 100,000 samples of 30 mice
- Build a distribution of sample means
- Width of this distribution is the true uncertainty of the mean

\[ \sigma_m = \frac{\sigma}{\sqrt{n}} = 0.9 \text{ g} \]

Real experiment
- 30 mice
- Measure body mass:
  9.9, 14.9, ..., 33.8 g
- Find standard error

\[ SE = \frac{SD}{\sqrt{n}} = 0.87 \text{ g} \]

*SE is an approximation of \( \sigma_m \)*
Sampling distribution of a proportion

- Width of the sampling distribution of a proportion
  \[ \sigma_R = \sqrt{\frac{p(1-p)}{n}} \]

- Replace an unknown population parameter, \( p \), with the observed estimator, \( \hat{p} \)
  \[ SE_R = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \]

- Standard error of a proportion
  \( SE_R \) estimates the width of the sampling distribution

- Approximate 95% CI is \( 1.96 \times SE_R \)

- However, this doesn’t work for small \( n \), or when proportion is close to 0 or 1
Example in R using prop.test

\[\text{prop.test(1, 10)}\]

1-sample proportions test with continuity correction

data: 1 out of 10, null probability 0.5
X-squared = 4.9, df = 1, p-value = 0.02686
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
0.005242302 0.458846016
sample estimates:
p
0.1
Confidence intervals of a proportion

- Consider survival experiment
  - take 10 mice
  - infect with something nasty
  - apply treatment
  - count survival proportion over time

- We need errors of proportion!
Confidence intervals of a proportion

- Consider survival experiment
  - take 10 mice
  - infect with something nasty
  - apply treatment
  - count survival proportion over time

- 95% CIs using Wald method
- The bigger sample, the smaller error

- Even when \( \hat{p} = 0 \), error allows for non-zero proportion
- We have zombie mice!
Beware of small samples!

- When you count things, small samples are not very good
- Consider a small number $n = 10$

|                  | Count        | Correlation $r = 0.73$ | Proportion $\hat{S} = 3$
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>95% CI</td>
<td>[4.8, 18.4]</td>
<td>[0.19, 0.93]</td>
<td>[0.10, 0.61]</td>
</tr>
<tr>
<td>Half CI as a fraction of the value</td>
<td>68%</td>
<td>51%</td>
<td>71%</td>
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</table>

- When you do counts, you need $n > 30$
Bootstrapping
## Bootstrapping

- Versatile technique used when
  - distribution of the estimator is complicated or unknown
  - for power calculations
- Approximate sampling distribution from one sample only
- Use random resampling *with replacement*

<table>
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<tr>
<th>19.4</th>
<th>18.2</th>
<th>11.5</th>
<th>17.2</th>
<th>25.7</th>
<th>19.2</th>
<th>21.5</th>
<th>16.7</th>
<th>15.6</th>
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<td>21.5</td>
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</tr>
</tbody>
</table>

- Repeat this many times (e.g. $10^5$) and collect all means
- Build the bootstrap distribution of the mean

$M = 18.6$  
$M = 18.8$ 
$M = 17.2$ 
$M = 16.2$ 
$M = 18.5$ 
$M = 19.4$
Bootstrapping

95% CI on the population mean
- from $t$-distribution [15.7, 21.5]
- from bootstrapping [16.3, 21.2]

This is not a sampling distribution, it only approximates it
Confidence intervals in R

Most statistical test functions in R provide with confidence intervals

<table>
<thead>
<tr>
<th>Quantity</th>
<th>R function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>t.test</td>
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<tr>
<td>Median</td>
<td>wilcox.test</td>
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<tr>
<td>Count</td>
<td>poisson.test</td>
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<td>Correlation</td>
<td>cor.test</td>
</tr>
<tr>
<td>Proportion</td>
<td>prop.test</td>
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</tbody>
</table>
> prop.test(12, 87)

1-sample proportions test with continuity correction

data: 12 out of 87, null probability 0.5
X-squared = 44.184, df = 1, p-value = 2.989e-11
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
  0.07637419 0.23243382
sample estimates:
    p
0.137931

# store test result in a variable
> p <- prop.test(12, 87)
# extract confidence intervals from object p
> ci <- p$conf.int
# ci is a vector of two elements: lower and upper CI limit
> ci[1]
[1] 0.07637419
> ci[2]
[1] 0.2324338
Hand-outs available at
https://dag.compbio.dundee.ac.uk/training/Statistics_lectures.html